

Problem 1 (11.2 #17). Determine whether the given boundary value problem is self-adjoint:

$$\begin{cases} (1+x^2)y'' + 2xy' + y = \lambda(1+x^2)y \\ y(0) - y'(1) = 0, \quad y'(0) + 2y(1) = 0 \end{cases}$$

Solution. First notice that we can rewrite the differential equation in the form:

$$-[-(1+x^2)y']' + y = \lambda(1+x^2)y$$

and so if the boundary conditions are separated we would be done... However, they are not separated (each condition involves both boundaries), so we must verify whether this problem is self-adjoint manually. First define the operator L as:

$$L[y] = [(1+x^2)y']' + y$$

We can also rewrite to boundary conditions to say: $y(0) = y'(1)$ and $y'(0) = -2y(1)$. Now we check to see if L satisfies the Lagrange identity: let u, v both be solutions of the boundary value problem above, then:

$$\begin{aligned} \langle L[u], v \rangle &= \int_0^1 \left\{ [(1+x^2)u']' + u \right\} v \, dx \\ &= \int_0^1 [(1+x^2)u']' v \, dx + \int_0^1 uv \, dx \\ &\stackrel{r=v}{=} \int_0^1 [(1+x^2)u']' v \, dx + \int_0^1 uv \, dx \\ &= (1+x^2)u'v \Big|_0^1 - \int_0^1 (1+x^2)u'v' \, dx + \int_0^1 uv \, dx \\ &\stackrel{r=(1+x^2)v'}{=} (1+x^2)u'v \Big|_0^1 - \int_0^1 (1+x^2)v'u' \, dx + \int_0^1 uv \, dx \\ &= (1+x^2)u'v \Big|_0^1 - \left((1+x^2)v'u \Big|_0^1 - \int_0^1 [(1+x^2)v']' u \, dx \right) + \int_0^1 uv \, dx \\ &= (1+x^2)u'v \Big|_0^1 - (1+x^2)v'u \Big|_0^1 + \int_0^1 [(1+x^2)v']' u \, dx + \int_0^1 uv \, dx \\ &= 2u'(1)v(1) - u'(0)v(0) - (2u(1)v'(1) - u(0)v'(0)) + \int_0^1 \left\{ [(1+x^2)v']' + v \right\} u \, dx \\ &\stackrel{B.C.s}{=} 2u'(1)v(1) - (-2u(1)v'(1)) - 2u(1)v'(1) + u'(1)(-2v(1)) + \int_0^1 \left\{ [(1+x^2)v']' + v \right\} u \, dx \\ &= 2u'(1)v(1) + 2u(1)v'(1) - 2u(1)v'(1) - 2u'(1)v(1) + \int_0^1 \left\{ [(1+x^2)v']' + v \right\} u \, dx \\ &= \int_0^1 \left\{ [(1+x^2)v']' + v \right\} u \, dx \\ &= \langle u, L[v] \rangle \end{aligned}$$

Thus the operator is self adjoint. □